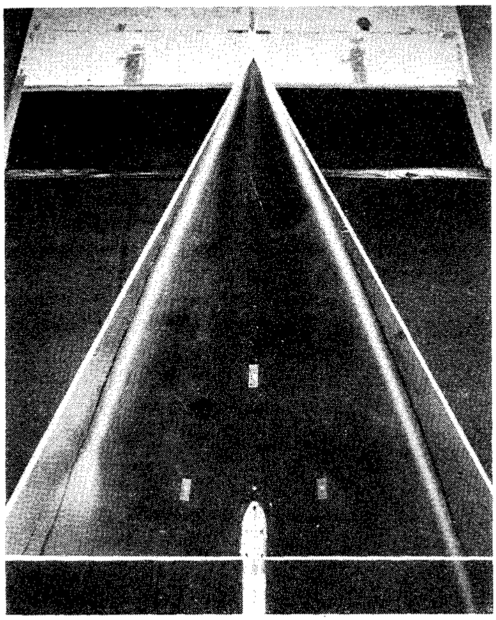

Fig. 3 Vortex flow pattern over a delta wing.

provements that would make the most difference in case of operation would be replacement of the current steam boiler with an off-the-shelf fully automatic boiler. A 1.5 boiler horse power (about the smallest available) would provide more than enough steam at adequate pressures (most units will operate up to at least 120 psi) while still being semiportable. Operation would only require setting the steam pressure and switching on.

On the nitrogen side, more pressure would be desirable, particularly for higher wind speeds. This is not possible with the current dewar, but high-pressure dewars capable of withstanding several hundred psi are available. Also the flexible tubing used for the nitrogen is not a cyrogenic part and stiffens at the reduced temperatures. Lines that remain flexible at the cyrogenic temperatures are available and will be used in future models. Another improvement is the tip for steam and nitrogen mixing. A new nozzle using peripheral injection of nitrogen around a central jet of steam is being tested, but higher nitrogen pressure than currently available is required for proper operation.

A new system constructed with these improvements should be easy to use and versatile. It is anticipated that hardware costs for the modified unit will be approximately $3000. In conclusion, the LN_2-steam system is a considerable improvement over existing methods of smoke production for flow visualization purposes.

Calculation of Initial Vortex Roll-Up in Aircraft Wakes

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Introduction

YATES,[1] in an interesting paper has recently given in closed form, the initial roll-up process of the vortex wake behind a lifting wing in symmetric flight. The intention of this Note is to rederive his results by using a Fourier sine series for the lift distribution instead of the Chebyshev polynomials he has used, and to complete the analysis by also considering nonsymmetric flight conditions. The results of Yates[1] and of this Note, therefore, form a quantitative step forward to Donaldson's basic hypothesis[2] that the initial vorticity shed between the local minima of the curve $|dF/dy|$ vs y will roll-up into a discrete vortex, by providing information on the inplane acceleration of the fluid particles in the wake. These results may be used, as pointed out by Yates as the first step in the use of the extended Betz theory for roll-up calculations.

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Analysis and Discussion

Following Yates, at time zero we have the following relations:

$$W(y) = (1/\pi) ⨍_{-1}^{1} \frac{F'(\eta)\,d\eta}{\eta - y} \tag{1}$$

$$V(y) = 0 \tag{2}$$

$$A(y) = \dot{V}(y) = -(1/\pi) ⨍_{-1}^{1} \left[\frac{W(y) - W(\eta)}{y - \eta} \right] \times \frac{F'(\eta)}{\eta - y}\, d\eta \tag{3}$$

where V, W are the velocity of the vortex filament (Y, Z) along the y and z axes (see Fig. 2, Ref. 1), $A(y)$ the initial inplane acceleration of the shed vorticity, $F(y)$ the normalized section lift or bound vortex strength, and the slash indicates the integrals to be of Cauchy principal value type. Also a prime denotes differentiation with respect to the function argument, and a dot differentiation with respect to time.

We expand the given $F(\eta)$ in a Fourier sine series with the following change of variables

$$\cos\theta = \eta \tag{4a}$$

$$\cos\phi = y \tag{4b}$$

so that

$$F(\theta) = \sum_{n=1}^{N} a_n \sin n\theta \tag{5}$$

and from Eq. (1)

$$W(\phi) = -(1/\pi) ⨍_{0}^{\pi} \sum_{n=1}^{N} n\, a_n \cos n\theta / (\cos\theta - \cos\phi)\, d\theta$$

$$= -\sum_{n=1}^{N} n\, a_n \sin n\phi / \sin\phi \tag{6}$$

The a_n can be calculated from

$$a_n = (2/\pi) \int_{0}^{\pi} F(\theta) \sin n\theta \, d\theta \tag{7}$$

by any standard numerical method.[4]

On substituting for F and W in Eq. (3) we find that

$$A(\phi) = (1/\pi) \sum_{m=1}^{N} m\, a_m \frac{\sin m\phi}{\sin\phi} \times ⨍_{0}^{\pi} \frac{\sum_{n=1}^{N} n\, a_n \cos n\theta}{(\cos\theta - \cos\phi)^2}\, d\theta$$

$$- (1/\pi)\int_0^{\pi} \sum_{m=1}^{N} m\, a_m \frac{\sin m\theta}{\sin\theta} \cdot \frac{\sum_{n=1}^{N} n\, a_n \cos n\theta}{(\cos\theta - \cos\phi)^2}\, d\theta$$

$$= \sum_{m=1}^{N} m\, a_m \frac{\sin m\phi}{\sin\phi} \sum_{n=1}^{N} n\, a_n I_{1,n}$$

$$- \sum_{m=1}^{N} \sum_{n=1}^{N} m\, n\, a_m\, a_n\, I_{m,n} \quad (8a)$$

where

$$I_{m,n} = (1/\pi)\int_0^{\pi} \frac{\sin m\theta \cos n\theta}{\sin\theta\,(\cos\theta - \cos\phi)^2}\, d\theta \quad (8b)$$

If the following trigonometric identities are used

$$\frac{\sin m\theta}{\sin\theta} = 2 \sum_{\substack{k=\xi+1\\ \xi+3,\ldots}}^{m-1} \cos k\theta + \xi \quad (9)$$

$$2 \sin k\theta \sin n\theta = \cos (k-n)\theta - \cos (k+n)\theta \quad (10a)$$

$$2 \cos k\theta \cos n\theta = \cos (k-n)\theta + \cos (k+n)\theta \quad (10b)$$

$$2 \sin k\theta \cos n\theta = \sin (k-n)\theta + \sin (k+n)\theta \quad (10c)$$

where

$$\xi = 0 \text{ for even } m \quad (11a)$$

$$= 1 \text{ for odd } m \quad (11b)$$

then

$$I_{m,n} = -(1/\sin^3\phi) \sum_{\substack{k=\xi+1\\ \xi+3,\ldots}}^{m-1}$$

$$\times [(k+n) \cos (k+n)\phi \sin\phi - \sin (k+n)\phi \cos\phi$$

$$+ |(k-n)| \cos (k-n)\phi \sin\phi - \sin |(k-n)|\phi \cos\phi] + \xi I_{1,n} \quad (12)$$

$$I_{1,n} = -(1/\sin^3\phi)[n \cos n\phi \sin\phi - \sin n\phi \cos\phi] \quad (13)$$

However, in numerical computations one may prefer to use the following recursive relation for $I_{m,n}$

$$I_{m,n} = \tfrac{1}{2}[I_{m-1,\,n+1} + I_{m-1,\,n-1} + I_{1,\,m+n-1} + I_{1,\,m-n-1}] \quad (14)$$

With all the $I_{m,n}$ known, the inplane acceleration can be calculated from Eq. (8).

These results, with Yates' assumption concerning multiple vortex roll-ups, may be used to obtain the number and strengths of the rolled-up vortex cores, for example, by using Rossow's analysis,[3] to determine aircraft wake hazard potential.

As an interesting example, consider an aircraft in symmetric flight carrying an elliptic span loading. Then it can be inferred that at $t=0$, the downwash in the wake is uniform and that the inplane acceleration is zero. This implies that the wake roll-up process is likely to be slow and hence the wake hazard may persist over larger distances. However, this loading gives a near uniform downwash field at the horizontal tail which could be a desirable feature from a stability and control point of view, as compared to multiple vortex cores forming in the wake and passing close to the horizontal tail. Multiple vortex cores have been suggested as a possible method of alleviating wake hazard potential by having several pronounced dips in the spanwise loading. Whereas this could be a useful solution during a landing approach, it is likely to give unacceptable drag penalities during take-off and climb, particularly when engine noise suppression measures have already started biting into available thrust.

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Derivatives of a Statically Reduced Stiffness Matrix with Respect to Sizing Variables

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An expression is derived for the first derivatives with respect to the sizing variables of a statically reduced stiffness matrix that is a nonlinear function of the sizing variables, where the unreduced stiffness matrix is a linear function of the sizing variables. In most methods of structural optimization (weight minimization) with flutter constraints the derivatives of the structural stiffness matrix with respect to the sizing variables (or design variables) β_i are used.[1] A structural representation satisfactory for aeroelastic analyses can be obtained from a finite element approach in which the elements are chosen such that the total stiffness matrix $[K(\beta_i)]$ is a linear function of the sizing variables:

$$[K(\beta_i)] = [K_0] + \sum_{i=1}^{n} \beta_i [\Delta K_i] \quad (1)$$

where $[K_0]$ represents an invariable stiffness corresponding to $[K(0)]$ and $[\Delta K_i]$ is a stiffness matrix associated with the sizing variable β_i. Under this condition the derivative of the stiffness matrix with respect to any sizing variable is a matrix that is independent of the sizing variables:

$$\frac{\partial}{\partial\beta_i} [K(\beta_i)] = [\Delta K_i] \quad (2)$$

Thus during an optimization procedure consisting of several steps, for each of which $[K(\beta_i)]$ must be evaluated according to Eq. (1), the derivatives of the stiffness matrix need to be evaluated only once.

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